The Turing Closure of an Archimedean Field

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Abstract

A BSS machine is δ-uniform if it does not use exact tests; such machines are equivalent (modulo parameters) to Type 2 Turing machines. We define a notion of closure related to Turing machines for archimedean fields, and show that such fields admit nontrivial δ-uniformly decidable sets iff they are not Turing closed. Then, the partially ordered set of Turing closed fields is proved isomorphic to the ideal completion of unsolvability degrees.

1 Introduction

In a previous paper [2], the authors have introduced a version of the BSS model of computability [1] in which exact tests are not allowed. Essentially, a BSS machine is δ-uniform iff its halting set and computed function do not change when the test for equality with 0 is replaced with a test for membership to an arbitrary ball around 0. A set is δ-uniformly semi-decidable iff it is the halting set of a δ-uniform BSS machine; as it turns out, such sets are always open.

There is a strict relation between δ-uniform computability and recursive analysis, i.e., Type 2 recursion theory; in fact, for any archimedean field the halting sets of δ-uniform BSS machines with coefficients in **T** (the field of Turing computable reals) or **Q** are exactly the halting sets of Type 2 Turing machines [2]. Thus, the restriction of δ-uniformity reduces the full power of the BSS model, making it closer to Turing machines (a deeper analysis of the relation between BSS and Type 2 decidability is pursued in [3]).

In this paper we solve a problem left open in [2], i.e., the characterization of the archimedean fields in which nontrivial δ-uniform decidable sets exist. Such sets, which must be clopen, do not exist in **R** by a simple connectedness argument; however, all archimedean fields are totally disconnected in the ball (Euclidean) topology, which calls for a more sophisticated approach. We shall introduce a notion of *Turing closure* of an archimedean field, and prove that a field possesses nontrivial δ-uniformly decidable sets iff it is not Turing closed. Moreover, if a function is δ-uniformly computable on a Turing closed field then it is rational over each of the connected components induced on the halting set by the reals. Note that the notion of Turing closure has an independent mathematical interest, as it deeply relates algebra and decidability theory; indeed, the proofs of the previous results require a nontrivial intertwining between topological, computational and algebraic arguments.

Finally, we relate Turing closures and degree theory [10] by proving the following result: the partially ordered set of Turing closed fields is isomorphic to that of ideals of unsolvability degrees. This gives a wealth of examples of Turing closed fields, and allows to prove some general theorems (such as the existence of minimal Turing closed fields above **T**) by translating known results from degree theory.

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2 The computation models

The problem of giving firm foundations to the notion of algorithm in a non-discrete realm has given rise to two opposing solutions:

- on one hand, following Turing [13], it is possible to study computations of Turing machines whose input tapes contain representations of real numbers (and that are allowed to output similar representations, as well): this approach is known as *Type 2 recursion theory* [15];
- on the other hand, one can take real numbers as primitive atomic objects, and study computational models that can operate on such objects, making exact computations and tests: this is precisely the viewpoint adopted in the BSS model [1].

In a previous paper [2], the authors introduced δ-uniform machines, i.e., (finite dimensional) BSS machines that do not perform exact comparisons: in other words, a δ-uniform machine can only decide whether two numbers are very close, but cannot decide whether they are truly equal or not.

In the following, we always consider machines working on an archimedean field R (recall that such fields are just subfields of $\mathbf{R}$, so we can freely identify the elements of R with real numbers; moreover, R is real, i.e., -1 is not a sum of squares).

2.1 δ-uniform BSS machines

A finite dimensional (normalized) BSS machine [1] is just a non-discrete version of a Random Access Machine: it takes inputs from R^m and produces outputs in R^n, using a state space whose registers contain elements of R. Informally, the program is described by a finite flowchart, where each non-final node is either a computation node or a branching node. Computation nodes have just one successor, and they are associated with a rational function of the state space into itself. Branching nodes have two successors, and the decision about which branch to take depends on whether the first coordinate x_1 of the state space is negative or not. The *coefficients* of a BSS machine are the coefficients of the rational functions appearing in its definition. A set which is the halting set of some BSS machine is called *semi-decidable*; if moreover its complement is also semi-decidable, we shall say that the set is *decidable*. Similarly, a partial function is *computable* if it is computed by some BSS machine. A set X is *semi-decidable relative to* Y if $X \cap Y$ is semi-decidable; it is *decidable relative to* Y if both $X \cap Y$ and $X^c \cap Y$ are semi-decidable.

A δ-uniform machine is a BSS machine in which a successful negativity test implies that the argument is strictly negative, while an unsuccessful test just claims that the argument was positive or in a neighbourhood of 0.

Formally, given a BSS machine M and a $\delta \geq 0$ (called a *threshold*), we define the δ-computing endomorphism much as in the classical case [1], but substituting the test case as follows [2]:

$$\langle q, \boldsymbol{x} \rangle \quad \mapsto \quad \begin{cases} \langle \beta^-(q), \boldsymbol{x} \rangle & \text{if } x_1 < -\delta \\ \langle \beta^+(q), \boldsymbol{x} \rangle & \text{if } x_1 \geq -\delta \end{cases} \qquad \text{if } q \text{ is a branching node.}$$

This induces a δ-halting set (denoted by Ω_M^δ) and a δ-computed function φ_M^δ.

Definition 1 A BSS machine M is δ-*uniform* if and only if $\Omega_M^\delta = \Omega_M$ and $\varphi_M^\delta = \varphi_M$ for all $\delta \in (0, 1)$.

The definition of δ-uniformity is the mathematical formalization of the fact that the threshold is not known to the programmer. The notions of (semi-)decidable set and of computable function carry over to the δ-uniform case[1].

[1]The choice of the interval $(0, 1)$ is immaterial in this definition: it is easy to see that a set (function) which is

2.2 Type 2 Turing machines

The tape of an ordinary Turing machine is nonblank only on a finite number of cells, at any moment of a computation. Thus, in order to allow elements of R to be taken into consideration, one slightly generalizes the notion of machine. A (deterministic Type 2 [16]) Turing machine consists of

1. a finite number of read-only one-way input tapes (possibly none), each containing at start an infinite string belonging to $\{\bar{1}, 0, 1, .\}^\omega$ and describing an element of R *via* its signed binary digit representation;

2. a finite number of write-only one-way output tapes (possibly none), on which the machine is supposed to write representations of elements of R;

3. some other work tapes, initially blank.

The finite control is defined as usual via a finite set of states and a transition function. The only differences with a standard Turing machine are the possibility of filling completely the input tapes, and of considering nonstopping machines as machines outputting elements of R.

The following theorem, proved in [2], gives an equivalence between δ-uniform and Type 2 decidability; the proof relies on dovetailing the emulation of a δ-uniform machine for all dyadic thresholds:

Theorem 1 Let $X \subseteq R^n$. Then X is δ-uniformly semi-decidable by a machine M with coefficients $\alpha_1, \dots, \alpha_r$ iff there exist a Type 2 Turing machine M' with $n + r$ input tapes such that for all $\langle x_1, \dots, x_n \rangle \in R^n$

$$\langle x_1, \dots, x_n \rangle \in X \iff M' \text{ halts on input } \langle x_1, \dots, x_n, \alpha_1, \dots, \alpha_r \rangle.$$

3 Turing extensions

As we already mentioned, every ordered archimedean field is (isomorphic to) an ordered subfield of $\mathbf{R}$ [14]: in the sequel, we shall use this fact and thus consider such fields simply as subfields of $\mathbf{R}$. An abstract theory of Turing closure (also for more general fields) is of course developpable along these lines as long as the elements of the involved fields are representable as infinite binary strings; the construction described here, in fact, should be more properly called *real* Turing closure. We leave these considerations to future work: for the rest of the paper, the word “field” will denote subfields of $\mathbf{R}$.

Definition 2 Let R be a field, and $R \subseteq F$ a field extension. An element $a \in F$ is said to be *Turing over* R iff there are $n \in \mathbf{N}$, $\boldsymbol{b} \in R^n$ and a Turing machine M (with n input tapes) such that $M(\boldsymbol{b}) = a$. If every $a \in F$ is Turing over R, then $R \subseteq F$ is said to be a *Turing extension* of R. A field R is *Turing closed* iff it does not have any proper Turing extension. The *Turing closure* of a field R is the intersection of all Turing closed fields containing R.

In other words, an element a is Turing over R if there is a Turing machine that outputs a, using a finite number of elements of R as input. Some easy facts can be remarked: each element of a field is Turing over that field (just using a Turing machine with one single input tape, on which the element itself is written). Moreover, $\mathbf{R}$ is clearly Turing closed (indeed, it is the largest Turing closed field). We also note that, as any familiar closure operator, Turing closure is monotone and idempotent. Finally, since $\mathbf{Q}$ is contained in any field and the Turing closure of $\mathbf{Q}$ is $\mathbf{T}$, we obtain that $\mathbf{T}$ (which has countable transcendence degree) is contained in every Turing closed field; as a consequence, the transcendence degree of a Turing closed field is equal to its cardinality.

δ-uniformly semi-decidable (computable) w.r.t. $(0, 1)$ is also δ-uniformly semi-decidable (computable) w.r.t. the whole set of positive elements.

Proposition 1 The Turing closure of R is given exactly by the set T of all reals that are Turing over R.

Proof. We just have to show that T is a Turing closed field. If $a, a' \in T$ then there are $\boldsymbol{b} \in R^n$, $\boldsymbol{b}' \in R^{n'}$ and Turing machines M, M' such that $M(\boldsymbol{b}) = a$ and $M'(\boldsymbol{b}') = a'$. But then there is a machine M'' with $n + n'$ input tapes that on input $\langle \boldsymbol{b}, \boldsymbol{b}' \rangle$ computes internally a, a' and writes the sum $a + a'$ on the output tape. Analogously for the other operations. Turing closedness can be easily shown by a suitable composition of Turing machines. ∎

4 Turing extension vs. algebraic extensions

In this section we prove that if a real number α is algebraic over a certain field R, then it can be produced by a Turing machine with inputs in that field; thus, α is also Turing over R, and as a consequence Turing closed fields are real closed [9]. Moreover, we show that there is a Turing machine M that "loses" a linear number of digits of its input $\boldsymbol{a}$ in writing α.

We firstly quote the following result from [5]; it is concerned with the convergence speed of Newton's method for finding the roots of a polynomial:

Lemma 1 Let $p(x) \in \mathbf{R}[x]$ be a polynomial, and α be a root of p; define the following function

$$T_p(x) = x - \frac{p(x)}{p'(x)},$$

which is Newton's rational transform. Then, there exists an interval I and an $\varepsilon \in (0, 1)$ such that the following statements hold:

- T_p is defined on every point of I, and $T_p(a) \in I$ for all $a \in I$;
- for all $z \in I$ and every n, $|T_p^n(z) - \alpha| \leq (1 - \varepsilon)^n$.

We now note that rational functions can be computed losing a constant number of digits:

Lemma 2 Let $f(x) \in R(x)$ be a rational function, and C be a compact subset of $\mathbf{R}$ on which f is defined. Then, there exist $a_1, \ldots, a_n \in R$, a Turing machine M with $n + 1$ input tapes, and an integer constant c such that

- for every $a \in C$, the machine M on input $\langle a_1, \ldots, a_n, a \rangle$ produces $f(a)$ as output;
- for every $k \geq c$, k bits of output are produced after reading at most $k - c$ bits of input.

Proof. Clearly there exist an $n \in \mathbf{N}$ and a rational function $g \in \mathbf{Q}(x_1, \ldots, x_n, x)$ such that $f(x) = g(a_1, \ldots, a_n)(x)$, with $a_i \in R$.

We use induction on the structure of g: if g is a variable or a constant, then $c = 0$. If $g = h_1 h_2$, we obtain by induction constants c_i for $h_i(a_1, \ldots, a_n)(x)$. Then, for every $a \in C$ by the bounds given in [4] we have that

$$\max\{c_1, c_2\} + 6 + \log(\max\{1, |h_1(a_1, \ldots, a_n)(a)|, |h_2(a_1, \ldots, a_n)(a)|\})$$

additional digits of the inputs are sufficient in order to output $g(a_1, \ldots, a_n)(a)$. The last expression is a continuous function of a, which can be maximized over C by compactness. Thus, we obtain a constant c for g. The treatment of sum and inversion is analogous. ∎

Thus, we obtain the following:

Theorem 2 Let α be algebraic over R. Then there is an $\boldsymbol{a} \in R^n$ and a Turing machine M such that $M(\boldsymbol{a}) = \alpha$ and the number of bits of $\boldsymbol{a}$ used by M in order to output l bits of α is linear in l.

Proof. Consider the minimum polynomial $p(x) \in R[x]$ of α. By Lemma 1 there is an interval I of length smaller than 1 entirely contained in the basin of α and an $0 < \varepsilon < 1$ such that

$$\forall z \in I \ \forall n \in \mathbf{N} \quad |T_p^n(z) - \alpha| \leq (1-\varepsilon)^n.$$

Let now c be the constant given by Lemma 2 for $T_p(z)$ on the interval I, and $d = -\log(1-\varepsilon) > 0$. We note that since $T_p(z) \in I$ for all $z \in I$, we can apply Lemma 2 to the iterates of T_p, obtaining that the computation of T_p^n on k input bits guarantees at least $k - cn$ correct output bits. We write $[T_p^n(z)]_k$ for the result of evaluating $T_p^n(z)$ using k bits of the inputs (i.e., of the coefficients of $p(x)$ and of z). Then, we have

$$\begin{aligned} |[T_p^n(z)]_k - \alpha| &\leq |[T_p^n(z)]_k - T_p^n(z)| + |T_p^n(z) - \alpha| \\ &\leq 2^{-(k-cn)} + (1-\varepsilon)^n = 2^{-(k-cn)} + 2^{-dn}. \end{aligned}$$

In order to bound the last sum, one essentially equates the exponents, obtaining the following functions of the desired number l of output bits:

$$k(l) = \left\lceil (c+d)\left(\frac{c+l+1}{d}\right)\right\rceil \qquad n(l) = \left\lceil \frac{k(l)}{c+d} \right\rceil.$$

Note that $n(l) > 0$; moreover (omitting the explicit dependence on l),

$$\begin{aligned} cn - k &\leq \frac{ck}{c+d} + c - k = \frac{-kd}{c+d} + c \\ &\leq (c+d)\left(\frac{c+l+1}{d}\right)\frac{-d}{c+d} + c = -c - l - 1 + c < 0, \end{aligned}$$

so $cn < k$ for all $l \geq 0$. Now observe that

$$-nd + k - cn = -n(c+d) + k \leq -\frac{k}{c+d}(c+d) + k = 0,$$

so $-nd \leq -k + cn$, and thus $|[T_p^n(z)]_k - \alpha| \leq 2^{-k+cn+1}$. Finally,

$$\begin{aligned} -k + cn + 1 &\leq -k + c\left(\frac{k}{c+d} + 1\right) + 1 = \frac{-dk}{c+d} + c + 1 \\ &\leq \frac{-d}{c+d}(c+d)\left(\frac{c+l+1}{d}\right) + c + 1 = -c - l - 1 + c + 1 = -l, \end{aligned}$$

so $|[T_p^n(z)]_k - \alpha| \leq 2^{-l}$. ∎

Corollary 1 If α is algebraic over R, then it is Turing over R as well.

5 Topological preliminaries

We now proceed to prove a series of topological lemmata, which will be essential in showing the connection between δ-uniform computability and Turing closed fields.

Definition 3 Given a connected topological space T and a dense subspace $D \subseteq T$, for every set $A \subseteq D$ open in D we define

$$\widetilde{A} = (A \cup D^c)^\circ.$$

Of course, this construction depends both on T and D: they will be always clear from the context. Note that $D^c = T \setminus D$ is totally disconnected. The main idea is that $\widetilde{A}$ is the open set "induced" by A on T.

Lemma 3 Given a connected topological space T and a dense subspace $D \subseteq T$, the following properties hold for sets $A, B \subseteq D$ which are open in D:

(i) if $A \cap B = \varnothing$ then $\widetilde{A} \cap \widetilde{B} = \varnothing$;

(ii) if $A \subseteq B$, then $\widetilde{A} \subseteq \widetilde{B}$;

(iii) $A \subseteq \widetilde{A}$;

(iv) $\widetilde{A} \cap D = A$;

(v) A is dense in $\widetilde{A}$;

(vi) if $A \cap B = \varnothing$ and $A \cup B = D$ then $\widetilde{A}$ and $\widetilde{B}$ are regular open sets[2];

(vii) suppose T is locally connected and C is a component of $\widetilde{A}$; then, $\widetilde{C \cap D} = C$.

Proof. (i). $\widetilde{A} \cap \widetilde{B} = (A \cup D^c)^\circ \cap (B \cup D^c)^\circ = [(A \cup D^c) \cap (B \cup D^c)]^\circ = [(A \cap B) \cup D^c]^\circ = (D^c)^\circ = \varnothing$.

(ii). By monotonicity of the interior operator.

(iii). If $x \in A$, then there is an open set $U \subseteq T$ such that $x \in U \cap D \subseteq A$; since $U \cup D^c = (U \cup D^c) \cap (D \cup D^c) = (U \cap D) \cup D^c$, we have

$$x \in U = U^\circ \subseteq (U \cup D^c)^\circ = [(U \cap D) \cup D^c]^\circ \subseteq (A \cup D^c)^\circ = \widetilde{A}.$$

(iv). One side is trivial by (iii); for the other inclusion, $\widetilde{A} \cap D \subseteq (A \cup D^c) \cap D = A$.

(v). Given an open set $U \subseteq T$ that meets $\widetilde{A}$, we have that $(U \cap \widetilde{A}) \cap D \subseteq \widetilde{A} \cap D = A$ cannot be empty, because $(U \cap \widetilde{A})$ is a nonempty open set of T. Thus, every nonempty open set of $\widetilde{A}$ meets A.

(vi). First note that A and B are separated in T. Indeed, if there is an $x \in \bar{A} \cap B$ then for all open sets $U \ni x$ we would have $U \cap A \neq \varnothing$. This implies $(U \cap D) \cap A \neq \varnothing$, i.e., B would not be open. Thus, $\bar{A} \subseteq A \cup D^c$, and similarly $\bar{B} \subseteq B \cup D^c$.

Now, if x is internal to the closure of $\widetilde{A}$, then there is an open neighbourhood $U \ni x$ contained in the closure of $\widetilde{A}$. All points of U are accumulation points of $\widetilde{A}$, thus of A by (v). This implies $x \in U \subseteq \bar{A} \subseteq A \cup D^c$, which yields $x \in \widetilde{A}$. The other inclusion is trivial.

(vii). Let $B = C \cap D$. First of all,

$$\widetilde{B} = [(C \cap D) \cup D^c]^\circ = (C \cup D^c)^\circ \supseteq C^\circ = C$$

(recall that C is open by local connectedness of T). On the other hand, suppose by contradiction that $x \in \widetilde{B} \setminus C$. If there is a (without loss of generality connected) neighbourhood $U \ni x$ such that $U \cap D \subseteq C$ (and thus $U \cap D \subseteq C \cap D \subseteq \widetilde{A} \cap D = A$), we have

$$U = U^\circ \subseteq (U \cup D^c)^\circ = [(U \cap D) \cup D^c]^\circ \subseteq \widetilde{A},$$

which implies $C \subset C \cup U \subseteq \widetilde{A}$, contradicting the maximality of C. We conclude that every neighbourhood of x must contain points of $D \setminus C = D \cap C^c = (D^c \cup C)^c$, and $x \notin \widetilde{B} = (C \cup D^c)^\circ$. ∎

[2] An open set is *regular* if it is the interior of its own closure.

Lemma 4 Let T be a topological space, and A, B two disjoint regular open sets, whose union is dense in T. Then A and B have the same boundary $\partial A = \partial B = (A \cup B)^c$; moreover, if T is connected and A,B are nonempty, then the boundary is nonempty.

Proof. Let x be a boundary point of A, and U an open neighbourhood of x; since $A \cup B$ is dense, U meets $A \cup B$. Clearly, U contains points in $A \subseteq B^c$, since x is in the boundary of A. We prove that U also contains points in B (which implies that x is also a boundary point of B). By contradiction, suppose that U does not contain any point of B: then every point of U would either belong to A or to $B^c \cap A^c$. The points of U that fall in $B^c \cap A^c$ cannot be all accumulation points of A, for otherwise U would be included in the closure of A, and thus x would be in the interior of A, because A is regular. So, at least one point y of U belongs to $B^c \cap A^c$ and it is an accumulation point of B. But then U is an open neighbourhood of y, and y is an accumulation point of B, so there is a point of B in U: a contradiction.

Note that $(A \cup B)^c = A^c \cap B^c \supseteq \partial A \cap \partial B$, so the points of the (common) boundary belong to the complement of $A \cup B$; the converse inclusion follows because every $x \in (A \cup B)^c$ belongs to $\overline{A \cup B}$ by density of $A \cup B$; however, $x \notin A \cup B = (A \cup B)^\circ$, so $x \in \partial(A \cup B) \subseteq \partial A \cup \partial B$.

Finally, suppose that $\partial A = \partial B = \varnothing$; then $T = \bar{A} \cup \bar{B} = A \cup B$, which is impossible, if T is connected, unless one of the two sets is empty. ∎

6 Decidability and Turing closure

We are now in the position to prove that

Theorem 3 The following conditions are equivalent:

(i) there is an $n > 0$, an open subset $Z \subseteq R^n$ such that $\widetilde{Z}$ is connected, and a nonempty set $X \subset Z$ which is δ-uniformly decidable relative to Z;

(ii) there is a nonempty set $X \subset (0, 1) \cap R$ which is δ-uniformly decidable relative to $(0, 1) \cap R$.

(iii) there is an $\alpha \in \mathbf{R} \setminus R$ which is Turing over R;

(iv) there is an $\alpha \in \mathbf{R} \setminus R$ such that $\{x \in R \mid x < \alpha\}$ is δ-uniformly decidable.

Proof. (i)⇒(ii). Let $X' = Z \setminus X$. By applying Lemma 3 with $T = \widetilde{Z}$ and $D = Z$, we have that $Y = [X \cup (\widetilde{Z} \setminus Z)]^\circ$ and $Y' = [X' \cup (\widetilde{Z} \setminus Z)]^\circ$ are[3] regular disjoint subsets of $\widetilde{Z}$ whose union is dense in $\widetilde{Z}$. Thus, we can apply Lemma 4, and consider a point y in $\partial Y = \partial Y'$. Since there is an open ball $B \ni y$ entirely contained in $\widetilde{Z}$, we take $x \in B \cap X$ and $x' \in B \cap X'$ by density, and consider the path φ connecting x to x', parameterized by $\varphi(t) = t(x' - x) + x$, where $t \in [0, 1] \subseteq \mathbf{R}$: observe that $t \in R$ iff $\varphi(t) \in R^n$.

Note that $\varphi^{-1}(X) \cap (0, 1)$ and $\varphi^{-1}(X') \cap (0, 1)$ are δ-uniformly decidable relative to $(0, 1) \cap R$: one just has to compute the corresponding point of Z, and use the machine deciding X (X'). Moreover, both $\varphi^{-1}(X) \cap (0, 1)$ and $\varphi^{-1}(X') \cap (0, 1)$ are nonempty; otherwise, every neighbourhood of 0 in R would contain points of $\varphi^{-1}(X')$ (if we assume without loss of generality that $\varphi^{-1}(X) \cap (0, 1)$ is empty); thus, every neighbourhood of x in Z would contain points of X', contradicting the fact that X is open.

(ii)⇒(iii). We define a Turing machine working as follows: given a dyadic interval (l, r) containing some points of both X and X', and initially set to $(0, 1)$, we find the minimum $k > 0$ such that the set of $2^k - 1$ dyadics of the form $l + i(r - l)/2^k$, for $0 < i < 2^k$, intersects both X and X' (in order to decide membership to X and X' we use Theorem 1);

[3] Note that, in order to avoid ambiguities, we use the tilde notation only for $T = \mathbf{R}^n$ and $D = R^n$.

note that this minimization is terminating because the numbers of this form for all k are dense in (l, r), and thus must intersect both X and X', which contain open neighbourhoods (i.e., intervals) in $(l, r) \cap R$. Then, we find the first j such that $l + j(r - l)/2^k \in X$ but $l + (j+1)(r-l)/2^k \in X'$ (we exchange the rôle of X and X' if such a j does not exists), and restart the process on the interval $(l + j(r - l)/2^k, l + (j + 1)(r - l)/2^k)$, which certainly contains points of both X and X' (because they are open), and whose length is at most $|r - l|/2$. The sequence of intervals thus defined cannot converge to a point of R (by openness of X and X'); hence, it converges to some number $\alpha \in \mathbf{R} \setminus R$, whose signed binary digits can be increasingly output each time a new subinterval is found.

(iii)⇒(iv). Take the Turing machine M writing α and emulate it with a δ-uniform machine M'. Then, for every input a generate α with enough precision in order to decide whether $a < \alpha$ or $a > \alpha$ (the case $\alpha = a$ being impossible).

(iv)⇒(i). Take $Z = R$. ▮

The main application of the previous theorem is the following

Corollary 2 Let R be an archimedean field. There are nontrivial δ-uniformly decidable subsets of R^n iff R is not Turing closed.

In particular, there are no nontrivial decidable subsets of $\mathbf{T}^n$ or $\mathbf{R}^n$. We now prove some restrictions about the functions computed over Turing closed fields:

Theorem 4 Let M be a δ-uniform machine, and C a component of $\widetilde{\Omega_M}$. If R is Turing closed, then $\varphi_{M|C \cap R^n}$ is a rational function.

Proof. Let $f_{\boldsymbol{a}}$ be the rational function of the input computed by M on input $\boldsymbol{a}$, $B = C \cap R^n$, and suppose $\varphi_{M|B}$ is not the restriction of a rational function. This implies that for some rational function g the sets $X = \{\boldsymbol{a} \in B \mid f_{\boldsymbol{a}} = g\}$ and $B \setminus X$ are both nonempty. Note that $\widetilde{B} = C$ is connected by Lemma 3, and that X (hence $B \setminus X$) is δ-uniformly decidable relative to B. Indeed, consider $E = \mathbf{Q}(a_1, \ldots, a_r) \subseteq R$, the extension of $\mathbf{Q}$ generated by the coefficients of M. By the primitive element theorem [9] we can recode all constants appearing in the program of M as elements of $\mathbf{Q}(x_1, \ldots, x_s)[x]/\langle p(x)\rangle$, where $\langle p(x)\rangle$ is the principal ideal generated by a certain irreducible polynomial in $\mathbf{Q}(x_1, \ldots, x_s)$ and $s \le r$ (following the lines of [2]). We emulate the computation of M with a machine M' that also keeps track of the intermediate results of the computation of M under the form of polynomials (the variables now being the input) with coefficient in $\mathbf{Q}(x_1, \ldots, x_s)[x]/\langle p(x)\rangle$; when M stops, the rational function computed can be tested exactly against g (also g can be coded, since its coefficients belong to E). By Theorem 3, R is not Turing closed. ▮

This implies, in particular, that the only total functions that are δ-uniformly computable on a Turing closed field are the rational functions. Moreover, Theorem 4 gives also a necessary condition, as explained by the following

Theorem 5 Let R be a field which is not Turing closed. Then, there exists a δ-uniform machine M computing a total function which is not rational.

Proof. We know, from Theorem 3, that there is some $\alpha \notin R$ such that $A = \{x \in R \mid x < \alpha\}$ is δ-uniformly decidable. Then the characteristic function $\chi_A : R \to \{0, 1\}$ (which is clearly not rational) is computable. ▮

7 Degrees of unsolvability and Turing closed fields

The notion of Turing closure gives a complete answer to the problem of the existence of sets δ-uniformly decidable over R. However, the existence of Turing closed fields besides

T and **R** is questionable, as well as the overall structure of the partially ordered set of Turing closed subfields of **R**.

In this section we deeply relate Turing closed fields and degrees of unsolvability from classical recursion theory. Essentially, we will show that the ideal completion of the partially ordered set of degrees is isomorphic to that of Turing closed subfields of **R**.

Consider a set $A \subseteq \mathbf{N}$; let μ_A be the least positive integer included in A (1 if A does not contain any positive integer), and let σ_A be either 1 or -1, depending on whether $0 \in A$ or not. Define

$$\rho(A) = \sigma_A \cdot \left(\mu_A - 1 + \sum_{\mu_A < i \in A} 2^{\mu_A - i} \right).$$

It should be clear that, for any non-dyadic real number a, there exists exactly one set A (which is neither finite nor cofinite) such that $\rho(A) = a$: this set will be denoted by $\rho^{-1}(a)$, and will be called the set *representing* a. We define the degree of a, denoted by $\operatorname{dg} a$, as the degree of unsolvability[4] of $\rho^{-1}(a)$ [11, 6]; moreover, we let $\operatorname{dg} a = 0$ for every dyadic rational a. Note that this representation and the representation used in the previous sections are equivalent in a computable way[5].

In the following, we shall consider ideals over the set of degrees; recall that a subset I of a sup-semilattice is an *ideal* if

(i) it is nonempty;

(ii) it is downward-closed, i.e., if $p \leq q \in I$ then also $p \in I$;

(iii) it is closed under binary suprema, i.e., if $p, q \in I$ then also $p \vee q \in I$.

It is easy to see that every nonempty set S is included in a minimum ideal $\widehat{S}$, the *ideal generated by* S, which satisfies

$$p \in \widehat{S} \iff p \leq \bigvee_{q \in T} q \text{ for some finite subset } T \text{ of } S.$$

In particular, the minimum ideal generated by a single element p is called the *principal ideal* of p.

In the following, the word "ideal" will always mean "ideal over the poset of degrees"; moreover, by abuse of notation, if $X \subseteq \mathbf{R}$ we shall write $\operatorname{dg} X$ for the set $\{\operatorname{dg} a \mid a \in X\}$. It is worth noting that:

Lemma 5 Let R be a subfield of **R**. A real number a is Turing over R iff $\operatorname{dg} a \in \widehat{\operatorname{dg} R}$.

Proof. We must prove that a is Turing over R iff there exists a finite $B \subseteq R$ such that $\operatorname{dg} a \leq \bigvee_{b \in B} \operatorname{dg} b$. There is a Turing machine M that outputs a using $b_1, \ldots, b_n \in R$ as input iff there is an oracle Turing machine M' (using n oracles) that decides membership to $\rho^{-1}(a)$, using $\rho^{-1}(b_1), \ldots, \rho^{-1}(b_n)$ as oracles (we can assume without loss of generality that all reals are non-dyadic; dyadic rationals can be easily ruled out, since they are all computable). This happens iff there is another machine M'' that decides $\rho^{-1}(a)$ using[6]

[4] We briefly recall, for sake completeness, that a set $A \subseteq \mathbf{N}$ is recursive in $B \subseteq \mathbf{N}$ iff there is an oracle Turing machine that decides membership to A using B as an oracle; this relation is a preorder on the subsets of **N**, and the equivalence classes induced by this preorder are called *degrees of unsolvability* [10]; they are of course a partially ordered set (the order relation being denoted by "$\leq$"), which possesses finite suprema denoted by "$\vee$"; the bottom element (corresponding to recursive sets) is denoted by 0. We write $\operatorname{dg} A$ for the degree of A.

[5] In fact, the degree of a real number is essentially independent of the representation chosen, as long as the representation of a number α can be converted to $\rho^{-1}(\alpha)$ back-and-forth in a recursive (non necessarily uniform) way. This happens for all classical representations—see [7].

[6] We denote with $A_1 + \cdots + A_n$ the coding of the disjoint union of the A_i's, represented as $\{\langle i, x \rangle \mid x \in A_i\}$, which is a subset of **N** by Cantor pairing.

$\rho^{-1}(b_1) + \cdots + \rho^{-1}(b_n)$ as an oracle, i.e., $\rho^{-1}(a)$ is recursive in $\rho^{-1}(b_1) + \cdots + \rho^{-1}(b_n)$. But

$$\mathrm{dg}(\rho^{-1}(b_1) + \cdots + \rho^{-1}(b_n)) = \mathrm{dg}\, b_1 \vee \cdots \vee \mathrm{dg}\, b_n,$$

hence the result. ▮

Our first result concerning the relation between ideals and Turing closed fields is the following

Theorem 6 For each Turing closed field R, $\mathrm{dg}\, R$ is an ideal of degrees. For each ideal I, the set $\mathrm{dg}^{-1} I = \{a \in \mathbf{R} \mid \mathrm{dg}\, a \in I\}$ is a Turing closed field.

Proof. To prove the first claim, just note that since R is Turing closed, $a \in R$ iff a is Turing over R, which happens iff $\mathrm{dg}\, a \in \widehat{\mathrm{dg}\, R}$ by Lemma 5.

For the second claim, we first prove that $\mathrm{dg}^{-1} I$ is a field (we just consider the case of multiplication; for addition and inverses the situation is analogous). Suppose that $a, b \in \mathrm{dg}^{-1} I$, and note that if a, b or ab are dyadic, the proof trivializes. Since I is an ideal, $\mathrm{dg}\, a \vee \mathrm{dg}\, b \in I$, and so there is an element c in $\mathrm{dg}^{-1} I$ that codes the set $\rho^{-1}(a) + \rho^{-1}(b)$. Multiplication is computable in positive notation for non-dyadics, so there is a machine that outputs ab using c as input; this proves that $ab \in \mathrm{dg}^{-1} I$. Finally, if a is Turing over $\mathrm{dg}^{-1} I$ then, by Lemma 5, $\mathrm{dg}\, a$ belongs to the ideal generated by $\mathrm{dg}(\mathrm{dg}^{-1} I)$, which is equal to I since dg is surjective, and so $a \in \mathrm{dg}^{-1} I$. ▮

Thus, we obtain the following corollary, showing that ideal generation is the degree-theoretical counterpart of Turing closure:

Corollary 3 Let R be an archimedean field and T be its Turing closure. Then $\widehat{\mathrm{dg}\, R} = \mathrm{dg}\, T$.

Proof. Since $\mathrm{dg}\, R \subseteq \mathrm{dg}\, T$ and $\mathrm{dg}\, T$ is an ideal by Theorem 6, $\widehat{\mathrm{dg}\, R} \subseteq \mathrm{dg}\, T$. Moreover, since $R \subseteq \mathrm{dg}^{-1}(\widehat{\mathrm{dg}\, R})$ and the latter is Turing closed (again by Theorem 6), $T \subseteq \mathrm{dg}^{-1}(\widehat{\mathrm{dg}\, R})$. ▮

Finally, we conclude by proving that

Theorem 7 The map dg is an isomorphism between the poset of Turing closed fields and the poset of ideals.

Proof. Note that dg and dg^{-1} restrict to functions between Turing closed fields and ideals by Theorem 6. Since their monotonicity is trivial, and $\mathrm{dg}(\mathrm{dg}^{-1} I) = I$ by surjectivity of dg, the second claim can be proved by showing that $\mathrm{dg}^{-1}(\mathrm{dg}\, T) \subseteq T$ for every Turing closed field T (the other inclusion being obvious). But $a \in \mathrm{dg}^{-1}(\mathrm{dg}\, T)$ implies $\mathrm{dg}\, a \in \mathrm{dg}\, T = \widehat{\mathrm{dg}\, T}$, and by Lemma 5 this happens iff a is Turing over T, i.e., $a \in T$. ▮

8 The complete lattice of Turing closed fields

Theorem 7 establishes an important relation between the structure of Turing closed fields and the poset of degrees of unsolvability; we can thus inherit many of the results so far obtained in the theory of degrees interpreting them as results on Turing closed fields. We just sketch some examples in this direction.

A theorem of Spector [12] proves that there exist minimal degrees; this yields immediately the following

Theorem 8 There exists a minimal Turing closed field, i.e., a Turing closed field whose only Turing closed (proper) subfield is $\mathbf{T}$.

Spector's result about the existence of minimal degrees is actually a special case of a more general theorem, due to Lachlan and Lebeuf [8], proving that every countable sup-semilattice with a least element is isomorphic to a countable ideal of degrees. This implies that:

Theorem 9 The ideal completion of every countable sup-semilattice with a least element is isomorphic to the poset of Turing closed subfields of some Turing closed field.

Call a Turing closed field R *principal* iff $\mathrm{dg}\,R$ is a principal ideal, i.e., if $R = \{a \in \mathbf{R} \mid \mathrm{dg}\,a \leq \mathrm{dg}\,r\}$ for some $r \in \mathbf{R}$. Principal fields are always countable, because every degree lies above a countable number of degrees [10].

Spector [12] proved that, for every countable ideal I, there exist two degrees $\mathfrak{a}_1, \mathfrak{a}_2$ such that $I = \{\mathfrak{b} \mid \mathfrak{b} < \mathfrak{a}_1 \text{ and } \mathfrak{b} < \mathfrak{a}_2\}$; from this we obtain:

Theorem 10 Let R be a countable Turing closed field. Then, one of the following holds:

(i) there are two principal Turing closed fields R_1, R_2 such that $R = R_1 \cap R_2$;

(ii) there is some $r \in \mathbf{R}$ such that $R = \{a \mid \mathrm{dg}\,a < \mathrm{dg}\,r\}$.

Proof. Consider the ideal $\mathrm{dg}\,R$: Spector's theorem says that either $\mathrm{dg}\,R$ is the intersection of two principal ideals (in the case that $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are incomparable), or there is some real number r satisfying the second condition. ∎

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